1895. The effect of weld residual stress on the free vibrational characteristics of cylindrical shell through the analytical method

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(Received 18 June 2015; received in revised form 8 December 2015; accepted 13 December 2015)

Abstract. The effect of weld residual stress on the free vibrational characteristics of cylindrical shell is investigated. Motion equations of cylindrical shell with weld residual stress are established based on Flügge theory, the interaction between weld residual stress and displacements is investigated. The analytical method is applied to calculate the vibrational mode. Weld residual stress can induce the variation of free vibrational characteristics. The amplitude mainly effects the variation magnitude of natural frequency and mode shape, and the distribution dose on the variation trend.

Keywords: weld residual stress, cylindrical shell, vibrational characteristics, natural frequency, mode shape.

Nomenclature

a	Cylindrical shell radius
h	Cylindrical shell thickness
l	Cylindrical shell length
x	Axial direction
φ	Circumferential direction
r	Radial direction
u	Axial displacement
v	Circumferential displacement
w	Radial displacement
z	Element normal direction
σ_x^r	Axial weld residual stress
σ_φ^r	Circumferential weld residual stress
ε_x	Strain at the middle plane in the x axis
ε_φ	Strain at the middle plane in the φ axis
N_x^r	Normal forces caused by weld residual stress along the x axis
N_φ^r	Normal forces caused by weld residual stress along the φ axis
$N_{x,\varphi}^r$	Projection of the normal force N_x^r in the φ axis
$N_{x,z}^r$	Projections of the normal force N_x^r in the z axis
ΔN_x^r	Variation of N_x^r in the x axis
$\Delta N_{x,\varphi}^r$	Variation of N_x^r in the φ axis
$\Delta N_{x,z}^r$	Variation of N_x^r in the z axis
$\Delta N_{\varphi,x}^r$	Variation of N_φ^r in the x axis
ΔN_φ^r	Variation of N_φ^r in the φ axis
$\Delta N_{\varphi,z}^r$	Variation of N_φ^r in the z axis
N_x	Normal force in the x axis caused by the vibration
$N_{\varphi x}$	Shearing force in the x axis caused by the vibration

N_φ	Normal force in the φ axis caused by the vibration
$N_{x\varphi}$	Shearing force in the φ axis caused by the vibration
Q_φ	Normal shearing force acting on the OA section caused by the vibration
M_x	Bending moment caused by the vibration around the x axis
M_φ	Bending moment caused by the vibration around the φ axis
$M_{\varphi x}$	Twisting moment caused by the vibration around the x axis
$M_{x\varphi}$	Twisting moment caused by the vibration around the φ axis
Q_x	Normal shearing force acting on the OC section caused by the vibration
μ	Poisson's ratio
ρ	Material density
t	Time
C_1, C_2, C_3	Weld residual stress and vibration displacement coupling terms

1. Introduction

The welding procedure is widely adopted in the process of submarine assembly and closure. During the welding uneven heating will result in weld residual stress, so it exists widely in the submarine hull [1, 2]. For the large structure, it is difficult to eliminate it though conventional methods. As the basic form of the submarine hull, it is necessary to evaluate the free vibrational behavior of the cylindrical shell with weld residual stress in order to prevent resonance fracture because of external dynamic loads.

Many scholars have done much investigations about weld residual stress, it is increasingly recognized that weld residual stress has the following characteristics [3-5]:

1) Weld residual stress includes tension and compression stress, they exist and balance in one component at the same time. Without the interference of external factors, weld residual stress will maintain stable in a long term.

2) The stress caused by the external load will overlay weld residual stress.

3) When the sum of stress caused by external load and weld residual stress exceeds the yield limit, plastic deformation will produce in the local area, and the distribution and amplitude of weld residual stress will change.

4) Weld residual stress mostly is in plane stress state, and it uniformly distributes along the thickness direction. It varies greatly along the thickness direction only in the thick section weld.

The research about the effect of weld residual stress on the free vibrational characteristics is fewer at present. Gao [6] found that weld residual stress has an influence on the natural frequency of the thin plate by experiments. He also proposed the formula of natural frequency of simple supported thin plate with weld residual stress [7]. The finite element method [8, 9] is used as a solution of vibrational mode of structures with weld residual stress, but it distributes mainly near the weld and varies drastically, so elements near the weld need to be divided very finely. For the large structure with long weld joints, it will spend lots of time on mesh generation.

In order to investigate the effect of weld residual stress on the free vibrational characteristics of cylindrical shell, the interaction between weld residual stress and displacements is investigated and motion equations of cylindrical shell weld residual stress are derived. And the analytical method is applied to calculate the vibrational mode. Finally, the effect of weld residual stress on the vibrational characteristics is discussed in detail.

2. Equations of motion

According to the above features of weld residual stress, the following assumptions are made:

1) Weld residual stress and the stress caused by vibration satisfy the linear superposition.

2) The release of weld residual stress does not occur during the vibration.

3) Weld residual stress maintains perpendicular to the cylindrical shell section during the

vibration.

4) Weld residual stress uniformly distributes along the thickness direction.

The geometry model of cylindrical shell is shown in Fig. 1. The thickness and radius are respectively denoted by a and h, h is considered small in comparison with other dimensions. x, φ and R respectively represent the axial, circumferential and radial directions. u, v and w respectively represent the axial, circumferential and radial displacements. Weld residual stress is generally resolved into two components in previous researches [1], which are respectively perpendicular and parallel to the weld joint. Due to the convenience of derivation, it is resolved into axial component σ_x^r and circumferential component σ_φ^r in the cylindrical coordinate here, which are functions of coordinates x and φ.

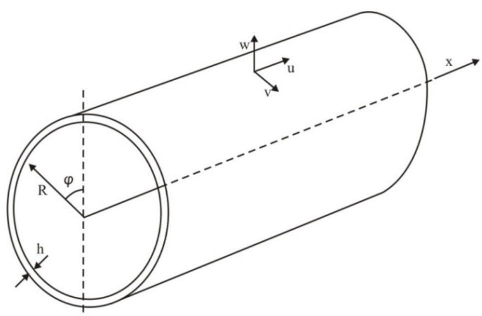


Fig. 1. The geometry model of cylindrical shell

Cut out a element of the shell. The z axis is taken perpendicular to the middle plane and positive in a downward direction. Forces acting on the element caused by weld residual stress are shown in Fig. 2. N_x^r and N_φ^r are normal forces caused by weld residual stress along the x and φ axes.

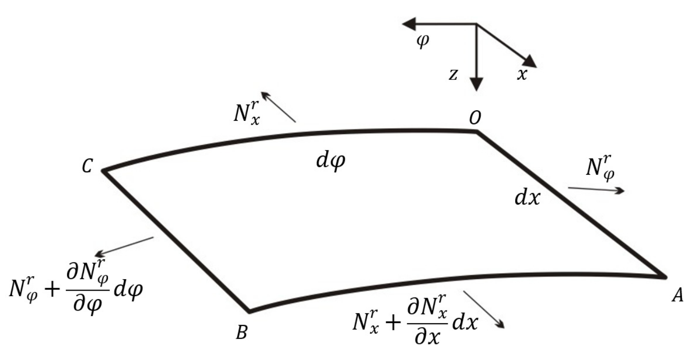


Fig. 2. Element body force caused by weld residual stress

The following will focus on forces caused by weld residual stress during the vibration. The element produces the strain due to the vibration, so lengths at distance z from the middle plane on OA and OC section become:

$$\begin{cases} l_{OC}^z = \left[\dfrac{a+z}{a} + \dfrac{a+z}{a}\left(\varepsilon_\varphi + \kappa_\varphi z\right)\right] a d\varphi, \\ l_{OA}^z = (1 + \varepsilon_x + \kappa_x z) dx, \end{cases} \tag{1}$$

where l_{AO}^z and l_{OC}^z respectively denote lengths at distance z from the middle plane on OA and OC section, ε_x and ε_φ are respectively strains at middle plane in the x and φ axes, κ_x and κ_φ respectively denote curvatures parallel to the xz and φz planes.

So areas of OA and OC section become:

$$\begin{cases} S_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} l_{OA}^z \, a d\varphi dz = h(1+\varepsilon_x) a d\varphi, \\ S_\varphi = \int_{-\frac{h}{2}}^{\frac{h}{2}} l_{OC}^z \, dx dz = \left[h\left(1+\varepsilon_\varphi\right) + \dfrac{h^3 \kappa_\varphi}{12a} \right] dx. \end{cases} \tag{2}$$

Flügge [9] proposed that in order to ensure the work done by the weld residual stress is zero, the normal force in the x axis during the vibration should be expressed as:

$$N_x^r = \sigma_x^r S_x = \sigma_x^r h(1+\varepsilon_x) a d\varphi. \tag{3}$$

N_x^r is no longer parallel to the x axis due to the vibration, it rotates around the φ and z axes (shown in Fig. 3), angles are respectively equal to $\partial v/\partial x$ and $\partial w/\partial x$.

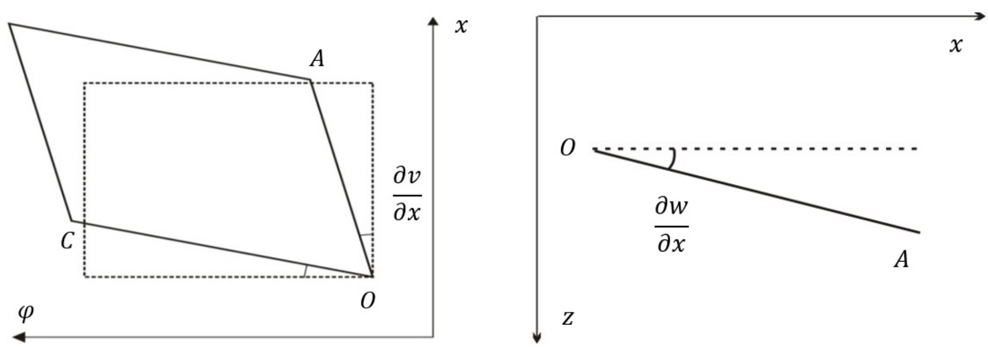


Fig. 3. Element body section angle

So projections of the normal force N_x^r in the φ and z axes can be expressed as ignoring the high order term:

$$\begin{cases} N_{x,\varphi}^r = N_x^r \dfrac{\partial v}{\partial x} = \sigma_x^r h(1+\varepsilon_x) \dfrac{\partial v}{\partial x} \approx \sigma_x^r h \dfrac{\partial v}{\partial x} a d\varphi, \\ N_{x,z}^r = N_x^r \dfrac{\partial w}{\partial x} = \sigma_x^r h(1+\varepsilon_x) \dfrac{\partial w}{\partial x} \approx \sigma_x^r h \dfrac{\partial w}{\partial x} a d\varphi, \end{cases} \tag{4}$$

where $N_{x,\varphi}^r$ and $N_{x,z}^r$ respectively denote projections of the normal force N_x^r in the φ and z axes.

When the element is static, N_x^r which is equal to $\sigma_x^r h a d\varphi$ is parallel to the x axis, and there is no projection in other axes. When the element vibrates, variation of N_x^r in three axes are:

$$\begin{cases} \Delta N_x^r = \sigma_x^r h(1+\varepsilon_x) a d\varphi - \sigma_x^r h a d\varphi = \sigma_x^r h \varepsilon_x a d\varphi, \\ \Delta N_{x,\varphi}^r = \sigma_x^r h \dfrac{\partial v}{\partial x} a d\varphi - 0 = \sigma_x^r h \dfrac{\partial v}{\partial x} a d\varphi, \\ \Delta N_{x,z}^r = \sigma_x^r h \dfrac{\partial w}{\partial x} a d\varphi - 0 = \sigma_x^r h \dfrac{\partial w}{\partial x} a d\varphi, \end{cases} \tag{5}$$

where ΔN_x^r, $\Delta N_{x,\varphi}^r$, and $\Delta N_{x,z}^r$ respectively denote variations of N_x^r in the x, φ and z axes. They both have the relationship with weld residual stress and displacement, which are called the coupling force.

Similarly, due to the strain caused by vibration, the normal force along the φ axis can be expressed as ignoring higher order term:

$$N_\varphi^r = \sigma_\varphi^r S_\varphi = \sigma_\varphi^r \left[h\left(1+\varepsilon_\varphi\right) + \frac{h^3 \kappa_\varphi}{12a} \right] dx \approx \sigma_\varphi^r h\left(1+\varepsilon_\varphi\right) dx. \tag{6}$$

During the vibration, N_φ^r rotates around the φ and z axes (shown in Fig. 4), angles are respectively equal to $\partial u / a\,\partial\varphi$ and $(\partial w/\partial\varphi - v)/a$.

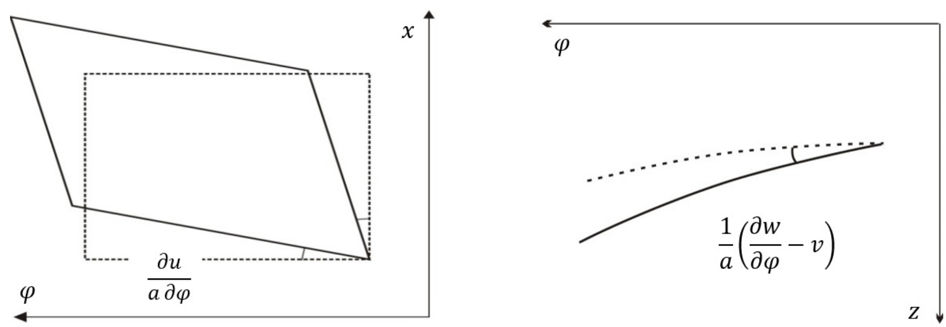


Fig. 4. Element body section angle

Similarly, when the element is static, N_φ^r equal to $\sigma_\varphi^r h dx$ is parallel to the x axis, and there is no projection in other axes. When the element vibrates, variations of N_φ^r in three axes are:

$$\begin{cases} \Delta N_{\varphi,x}^r = \sigma_\varphi^r h \dfrac{\partial u}{a\,\partial\varphi} dx - 0 = \sigma_\varphi^r h \dfrac{\partial u}{a\,\partial\varphi} dx, \\ \Delta N_\varphi^r = \sigma_\varphi^r h\left(1+\varepsilon_\varphi\right)dx - \sigma_\varphi^r h dx = \sigma_\varphi^r h \varepsilon_\varphi dx, \\ \Delta N_{\varphi,z}^r = \sigma_\varphi^r \dfrac{h}{a}\left(\dfrac{\partial w}{\partial\varphi} - v\right) dx - 0 = \sigma_\varphi^r \dfrac{h}{a}\left(\dfrac{\partial w}{\partial\varphi} - v\right) dx, \end{cases} \tag{7}$$

where $\Delta N_{\varphi,x}^r$, ΔN_φ^r and $\Delta N_{\varphi,z}^r$ respectively denote variations of N_φ^r in the x, φ and z axes, they are also coupling forces.

Since weld residual stress always remains perpendicular to the shell section and uniformly distributes along the thickness direction, it doesn't produce new couple bending and twisting moments.

Now establish dynamic equilibriums in three axes. Assume that there is no external load acting on the cylindrical shell. Besides N_x and $N_{\varphi x}$ caused by the vibration, there are coupling forces ΔN_x^r and $\Delta N_{x,\varphi}^r$ in the x axis. So the equilibrium in this direction can be expressed as:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{\varphi x}}{a\,\partial\varphi} + \frac{\partial \Delta N_x^r}{\partial x} + \frac{\partial \Delta N_{\varphi,x}^r}{a\,\partial\varphi} = \rho h \frac{\partial^2 u}{\partial t^2}, \tag{8}$$

where N_x and $N_{\varphi x}$ respectively denote the normal force and shearing force along the x axis caused by the vibration.

Next establish the equilibrium in the φ direction. Besides N_x, $N_{x\varphi}$ and Q_φ, there are coupling forces ΔN_φ^r and $\Delta N_{x,\varphi}^r$. So the equilibrium in this direction can be expressed as:

$$\frac{\partial N_\varphi}{a\,\partial\varphi} + \frac{\partial N_{x\varphi}}{\partial x} + \frac{Q_\varphi}{a} + \frac{\partial \Delta N_\varphi^r}{a\,\partial\varphi} + \frac{\partial \Delta N_{x,\varphi}^r}{\partial x} = \rho h \frac{\partial^2 v}{\partial t^2}, \tag{9}$$

where N_φ and $N_{x\varphi}$ respectively denote the normal force and shearing force along the φ axis caused by the vibration, Q_φ denote the normal shearing force acting on the OA section caused by the vibration.

Then establish the equilibrium in the z direction. Besides Q_φ, Q_x and N_φ, there are coupling forces $\Delta N_{x,z}^r$ and $\Delta N_{\varphi,z}^r$. So the equilibrium in this direction can be expressed as:

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_\varphi}{a\,\partial\varphi}+N_\varphi+\frac{\partial \Delta N_{x,z}^r}{\partial x}+\frac{\partial \Delta N_{\varphi,z}^r}{a\,\partial\varphi}=\rho h\frac{\partial^2 w}{\partial t^2}, \tag{10}$$

where Q_x denote the normal shearing force acting on the OC section caused by the vibration.

Finally establish moment equilibriums in the x and φ directions. Because weld residual stress doesn't produce new bending and twisting moment, moment equilibriums can be expressed as according to Flügge shell theory [10]:

$$\frac{\partial M_\varphi}{a\,\partial\varphi}+\frac{\partial M_{x\varphi}}{\partial x}+Q_\varphi=0, \tag{11}$$
$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{\varphi x}}{a d\varphi}+Q_x=0, \tag{12}$$

where M_x and M_φ respectively are bending moments caused by normal stress around the x and φ axes, $M_{\varphi x}$ and $M_{x\varphi}$ respectively are twisting moments caused by shearing stress around the x and φ axes.

Forces and moments on the element caused by the vibration can be solved by Flügge theory [10]. Substitute Eqs. (5), (7), (11) and (12) into Eqs. (8)-(10), then free motion equations of the cylindrical shell with weld residual stress are:

$$\begin{cases} L_1(u,v,w)+C_1\left(u,v,w,\sigma_x^r,\sigma_\varphi^r\right)=\dfrac{\rho h}{D}\dfrac{\partial^2 u}{\partial t^2}, \\ L_2(u,v,w)+C_2\left(u,v,w,\sigma_x^r,\sigma_\varphi^r\right)=\dfrac{\rho h}{D}\dfrac{\partial^2 v}{\partial t^2}, \\ L_3(u,v,w)-C_3\left(u,v,w,\sigma_x^r,\sigma_\varphi^r\right)=-\dfrac{\rho h}{D}\dfrac{\partial^2 w}{\partial t^2}, \end{cases} \tag{13}$$

where ρ is the material density, $D=Eh^3/12(1-\mu^2)$, E is the Young's modulus, μ is the Poisson's ratio and:

$$L_1(u,v,w)=\frac{12}{h^2}\frac{\partial^2 u}{\partial x^2}+\frac{6(1-\mu)}{a^2h^2}\frac{\partial^2 u}{\partial\varphi^2}+\frac{1-\mu}{2a^4}\frac{\partial^2 u}{\partial\varphi^2}+\frac{6(1+\mu)}{ah^2}\frac{\partial^2 v}{\partial x\,\partial\varphi}+\frac{12\mu}{ah^2}\frac{\partial w}{\partial x}$$
$$-\frac{\partial^3 w}{a\,\partial x^3}+\frac{1-\mu}{2a^3}\frac{\partial^3 w}{\partial x\,\partial\varphi^2},$$
$$L_2(u,v,w)=\frac{6(1+\mu)}{ah^2}\frac{\partial^2 u}{\partial\varphi\,\partial x}+\frac{12}{a^2h^2}\frac{\partial^2 v}{\partial\varphi^2}+\frac{6(1-\mu)}{h^2}\frac{\partial^2 v}{\partial x^2}+\frac{3(1-\mu)}{2a^2}\frac{\partial^2 v}{\partial x^2}$$
$$+\frac{12}{a^2h^2}\frac{\partial w}{\partial\varphi}-\frac{3-\mu}{2a^2}\frac{\partial^3 w}{\partial x^2\,\partial\varphi},$$
$$L_3(u,v,w)=\frac{12\mu}{ah^2}\frac{\partial u}{\partial x}-\frac{\partial^3 u}{a\,\partial x^3}+\frac{1-\mu}{2a^3}\frac{\partial^3 u}{\partial x\,\partial\varphi^2}+\frac{12}{a^2h^2}\frac{\partial v}{\partial\varphi}-\frac{3-\mu}{2a^2}\frac{\partial^3 v}{\partial x^2\,\partial\varphi}+\frac{12w}{a^2h^2}+\frac{\partial^4 w}{\partial x^4}$$
$$+\frac{2\,\partial^4 w}{a^2\,\partial x^2\,\partial\varphi^2}+\frac{\partial^4 w}{a^4\,\partial\varphi^4}+\frac{w}{a^4}+\frac{2}{a^4}\frac{\partial^2 w}{\partial\varphi^2},$$

$$C_1(u,v,w,\sigma_x^r,\sigma_\varphi^r)=\frac{h}{D}\left(\frac{\partial\sigma_x^r}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial\sigma_\varphi^r}{a^2\partial\varphi}\frac{\partial u}{\partial\varphi}+\sigma_x^r\frac{\partial^2 u}{\partial x^2}+\frac{\sigma_\varphi^r}{a^2}\frac{\partial^2 u}{\partial\varphi^2}\right),$$
$$C_2(u,v,w,\sigma_x^r,\sigma_\varphi^r)=\frac{h}{D}\left(\frac{\partial\sigma_\varphi^r}{a^2\partial\varphi}\frac{\partial v}{\partial\varphi}+\frac{\partial\sigma_x^r}{\partial x}\frac{\partial v}{\partial x}+\sigma_\varphi^r\frac{\partial^2 v}{a^2\partial\varphi^2}+\sigma_x^r\frac{\partial^2 v}{\partial x^2}+\frac{\partial\sigma_\varphi^r}{\partial\varphi}\frac{w}{a^2}+\frac{\sigma_\varphi^r}{a^2}\frac{\partial w}{\partial\varphi}\right),$$
$$C_3(u,v,w,\sigma_x^r,\sigma_\varphi^r)=\frac{h}{D}\left(-\frac{\partial\sigma_\varphi^r}{\partial\varphi}\frac{v}{a^2}-\frac{\sigma_\varphi^r}{a^2}\frac{\partial v}{\partial\varphi}+\frac{\partial\sigma_\varphi^r}{a^2\partial\varphi}\frac{\partial w}{\partial\varphi}+\frac{\partial\sigma_x^r}{\partial x}\frac{\partial w}{\partial x}+\sigma_x^r\frac{\partial^2 w}{\partial x^2}+\sigma_\varphi^r\frac{\partial^2 w}{a^2\partial\varphi^2}\right).$$

$C_1(u,v,w,\sigma_x^r,\sigma_\varphi^r)$, $C_2(u,v,w,\sigma_x^r,\sigma_\varphi^r)$ and $C_3(u,v,w,\sigma_x^r,\sigma_\varphi^r)$ are called as coupling terms of weld residual stress and vibration displacement, and denoted by C_1, C_2 and C_3 for simplification in the following. Weld residual stress is the function of the coordinates x and φ, so its partial derivative of spatial coordinates can't be ignored. Compared with classic motion equations of the cylindrical shell without weld residual stress, Eq. (13) includes coupling terms.

3. Analytical solution

The displacement of the cylindrical shell with simply supported edges can be represented by double trigonometric series as follow [10]:

$$\begin{cases} u=\sum_{\eta=1}^{M}\sum_{\varsigma=1}^{N}U_{\eta\varsigma}\cos(\varsigma\varphi)\cos(\eta\lambda x)\sin(\omega t), \\ v=\sum_{\eta=1}^{M}\sum_{\varsigma=1}^{N}V_{\eta\varsigma}\sin(\varsigma\varphi)\sin(\eta\lambda x)\sin(\omega t), \\ w=\sum_{\eta=1}^{M}\sum_{\varsigma=1}^{N}W_{\eta\varsigma}\cos(\varsigma\varphi)\sin(\eta\lambda x)\sin(\omega t), \end{cases} \tag{14}$$

where $\eta=\pi/l$, l is the length of the cylindrical shell, ω is the frequency, t is the time.

Substituting Eq. (14) into Eq. (13), we can obtain the following equations by making use of the orthogonality of trigonometric series:

$$\begin{aligned}&\left[-\frac{12}{h^2}(m\lambda)^2-\frac{6(1-\mu)}{a^2h^2}n^2-\frac{1-\mu}{2a^4}n^2\right]U_{nm}+\frac{6(1+\mu)}{ah^2}nm\lambda V_{nm}\\&\quad+\left[\frac{12\mu}{ah^2}m\lambda+\frac{(m\lambda)^3}{a}-\frac{1-\mu}{2a^3}m\lambda n^2\right]W_{nm}+K_1+\frac{\rho h\omega^2}{D}U_{nm}=0,\end{aligned} \tag{15}$$

$$\begin{aligned}&\frac{6(1+\mu)}{ah^2}nm\lambda U_{nm}-\left[\frac{12n^2}{a^2h^2}+\frac{6(1-\mu)}{h^2}(m\lambda)^2+\frac{3(1-\mu)}{2a^2}(m\lambda)^2\right]V_{nm}\\&\quad-\left[\frac{12n}{a^2h^2}+\frac{3-\mu}{2a^2}(m\lambda)^2n\right]W_{nm}+K_2+\frac{\rho h\omega^2}{D}V_{nm}=0,\end{aligned} \tag{16}$$

$$\begin{aligned}&\left[\frac{12\mu}{ah^2}m\lambda+\frac{(m\lambda)^3}{a}-\frac{1-\mu}{2a^3}m\lambda n^2\right]U_{nm}-\left[\frac{12n}{a^2h^2}+\frac{3-\mu}{2a^2}(m\lambda)^2n\right]V_{nm}\\&\quad-\left[\frac{12}{a^2h^2}+\left(m^2\lambda^2+\frac{n^2}{a^2}\right)^2+\frac{1}{a^4}-2\frac{n^2}{a^4}\right]W_{nm}+K_3+\frac{\rho h\omega^2}{D}W_{nm}=0,\end{aligned} \tag{17}$$

where:

$$K_1 = \int_0^{2\pi} \int_0^l C_1 \cos(n\varphi)\cos(m\lambda x)dxd\varphi, \quad K_2 = \int_0^{2\pi} \int_0^l C_2 \sin(n\varphi)\sin(m\lambda x)dxd\varphi,$$
$$K_3 = \int_0^{2\pi} \int_0^l C_3 \sin(n\varphi)\sin(m\lambda x)dxd\varphi.$$

When there is a girth weld on the cylindrical shell, the distribution of weld residual stress can be approximately considered to be axisymmetric due to the geometrical axisymmetry. In addition, tensile and compressive residual stress maintain equilibrium in the shell. So weld residual stress is suitable for expressing in the form of single triangular series:

$$\begin{cases} \sigma_x^r = \sum_{g=1}^{M} \sigma_g^{rx} \cos(g\lambda x), \\ \sigma_\varphi^r = \sum_{g=1}^{M} \sigma_g^{r\varphi} \cos(g\lambda x), \end{cases} \tag{18}$$

where $\sigma_g^{rx} = \frac{2}{l}\int_0^l \sigma_x^r \cos(g\lambda x)dx$, $\sigma_g^{r\varphi} = \frac{2}{l}\int_0^l \sigma_\varphi^r \cos(g\lambda x)dx$, σ_g^{rx} and $\sigma_g^{r\varphi}$ are amplitudes of weld residual stress, g is the positive integer not less than one.

Substitute Eq. (18) into $K1$, $K2$ and $K3$, then $M\times N$ equations are established by making use of Appendix Eqs. (A1)-(A4) and (A10)-(A13), which are expressed as matrix form:

$$(\mathbf{\Lambda} + \mathbf{R})\mathbf{X} = \mathbf{0}, \tag{19}$$

where $\mathbf{X} = \{\mathbf{U}_1 \cdots \mathbf{U}_{(m-1)\times N+n} \cdots \mathbf{U}_{M\times N}\}^T$, $\mathbf{U}_{(m-1)\times N+n} = \{U_{mn} \quad V_{mn} \quad W_{mn}\}^T$, $\mathbf{R} = \sum_{g=1}^{G} \mathbf{R}_g$, $\mathbf{R}_g$ and $\mathbf{\Lambda}$ are clearly shown in Appendix Eqs. (A18) and (A19).

If weld residual stress is not axisymmetric, it will be suitable for expressing in the form of double triangular series:

$$\begin{cases} \sigma_x^r = \sum_{g=1}^{G}\sum_{j=1}^{J} \sigma_{gj}^{rx} \cos(g\lambda x)\cos(j\varphi), \\ \sigma_\varphi^r = \sum_{g=1}^{G}\sum_{j=1}^{J} \sigma_{gj}^{r\varphi} \cos(g\lambda x)\cos(j\varphi), \end{cases} \tag{20}$$

where $\sigma_{gj}^{rx} = \frac{2}{\pi l}\int_0^l \int_0^{2\pi} \sigma_x^r \cos(g\lambda x)\cos(j\varphi)dxd\varphi$, $\sigma_{gj}^{r\varphi} = \frac{2}{\pi l}\int_0^l \int_0^{2\pi} \sigma_\varphi^r \cos(g\lambda x)\cos(j\varphi)dxd\varphi$, σ_g^{rx} and $\sigma_g^{r\varphi}$ are amplitudes of weld residual stress, g and j are positive integers not less than one.

Substitute Eq. (20) into $K1$, $K2$ and $K3$, then $M\times N$ equations by making use of Appendix Eqs. (A1)-(A17) similarly, which are expressed as matrix form:

$$(\mathbf{\Lambda} + \mathbf{R})\mathbf{X} = \mathbf{0}, \tag{21}$$

where:

$$\mathbf{X} = \{\mathbf{U}_1 \cdots \mathbf{U}_{(m-1)\times N+n} \cdots \mathbf{U}_{M\times N}\}^T, \quad \mathbf{U}_{(m-1)\times N+n} = \{U_{mn} \quad V_{mn} \quad W_{mn}\}^T,$$
$$\mathbf{R} = \sum_{g=1}^{G}\sum_{j=1}^{J} \mathbf{R}_{gj},$$

$\mathbf{R}_{gj}$ and $\mathbf{\Lambda}$ are clearly shown in Appendix Eqs. (A18) and (A20).

Eqs. (19) and (21) are linear systems of equations, the determinant of equation coefficient is set equal to zero, namely:

$$|\mathbf{\Lambda} + \mathbf{R}| = 0. \tag{22}$$

The eigenvalue and eigenvector can be obtained by solving Eq. (22). The existence of **R** definitely leads to the variation of free vibrational behavior.

4. Discussion

4.1. Natural frequency

Take the two edges simply supported cylindrical shell with weld residual stress for example, which geometry and material parameters are shown in Table 1.

There is a girth weld on the cylindrical shell. Assume that the distributions of weld residual stress at every φ section are identical, which are shown in Fig. 5.

Table 1. Cylindrical shell geometry and material parameters

Geometry parameter	Value	Unit	Material parameter	Value	Unit
Length	5	m	Young's modulus	2.1×10^{11}	N/m^2
Width	0.5	m	Poisson's ratio	0.3	
Thickness	0.014	m	Density	7860	kg/m^3

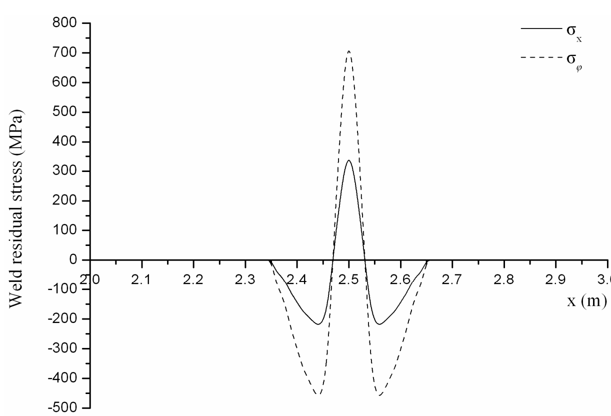


Fig. 5. Weld residual stress distribution No. 1

Firstly, discuss the effect of weld residual stress amplitude on the natural frequency. On the premise of maintaining the distribution (shown in Fig. 6), its amplitude is respectively equal to 100 %, 75 % and 50 %. The first ten natural frequencies under different amplitude solved by the analytical method are shown in Table 2.

Difference of natural frequency under every weld residual stress amplitude is shown in Fig. 6. It can be found that the amplitude of weld residual stress has a great impact on natural frequency, especially variation rate of the first order is up to 6.78 %. With increase of the amplitude, the variation magnitude of natural frequencies grows, but the variation trend basically remains the same. Nowadays high strength steel is widely used as submarine pressure hull. It has the high yield limit, so its amplitude of weld residual stress is higher than general structural steel. Consequently, weld residual stress has the greater impact on natural frequency.

Next the effect of weld residual stress distribution on natural frequencies will be discussed. Two kinds of weld residual stress distribution which have the same amplitude are shown in Figs. 5 and 7.

Table 2. The first ten natural frequencies under different weld residual stress amplitude

Order	Without weld residual stress (Hz)	With weld residual stress					
		100 % (Hz)	Difference (%)	75 % (Hz)	Difference (%)	50 % (Hz)	Difference (%)
1	52.07	48.55	–6.78	49.46	–5.01	50.36	–3.30
2	101.86	101.91	0.05	101.90	0.04	101.88	0.03
3	108.25	102.31	–5.48	103.98	–3.94	105.53	–2.51
4	128.24	131.21	2.32	130.49	1.75	129.75	1.18
5	137.15	138.02	0.63	137.81	0.48	137.59	0.32
6	181.54	180.93	–0.34	181.01	–0.29	181.13	–0.22
7	204.32	196.12	–4.01	198.67	–2.76	200.92	–1.66
8	211.28	214.66	1.60	213.88	1.23	213.06	0.84
9	229.78	230.27	0.21	229.81	0.01	229.54	–0.10
10	263.02	262.96	–0.02	262.98	–0.02	262.99	–0.01

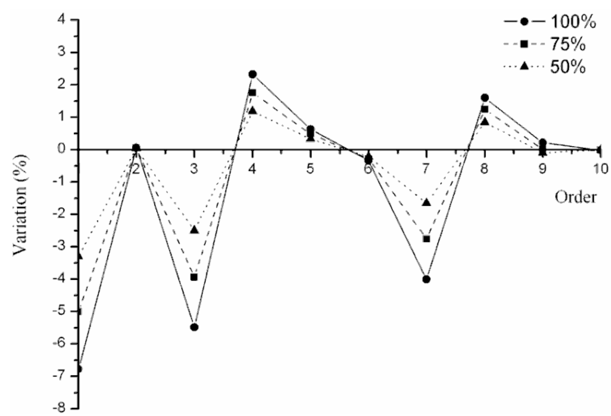


Fig. 6. The impact of weld residual stress amplitude on the natural frequency

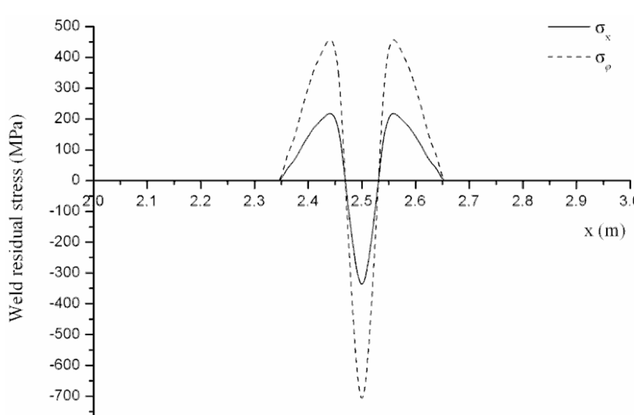


Fig. 7. Weld residual stress distribution No. 2

The first ten natural frequencies under different amplitude solved by the analytical method are shown in Table 3.

Difference of natural frequency under every kind of weld residual stress distribution is shown in Fig. 8. It can be found that different kind of weld residual stress distribution has different impact on natural frequencies. For example, distribution No. 1 makes the first-order natural frequency decrease, whereas distribution No. 2 makes it increase. Thus, the distribution of weld residual stress plays an important role in the variation trend of natural frequencies.

Table 3. The impact of weld residual stress distribution on natural frequencies

Order	Without weld residual stress (Hz)	With weld residual stress			
		No. 1 (Hz)	Difference (%)	No. 2 (Hz)	Difference (%)
1	52.07	48.55	–6.78	55.26	6.12
2	101.86	101.91	0.05	101.80	–0.05
3	108.25	102.31	–5.48	112.27	3.72
4	128.24	131.21	2.32	125.07	–2.47
5	137.15	138.02	0.63	136.27	–0.64
6	181.54	180.93	–0.34	182.99	0.80
7	204.32	196.12	–4.01	206.28	0.96
8	211.28	214.66	1.60	207.26	–1.90
9	229.78	230.27	0.21	233.74	1.72
10	263.02	262.96	–0.02	263.04	0.01

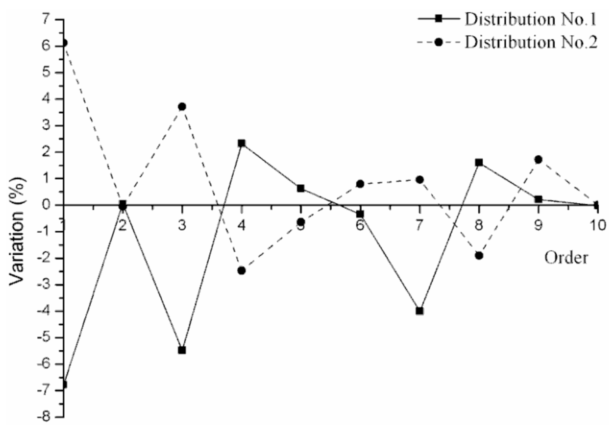
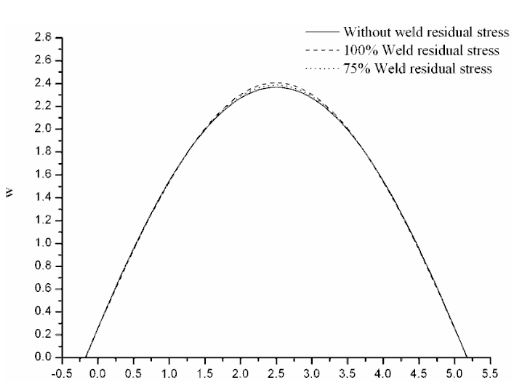


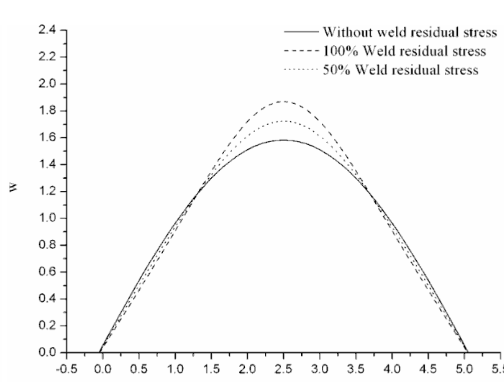
Fig. 8. The effect of weld residual stress distribution on natural frequencies

4.2. Mode shape

This section will focus on the impact of weld residual stress on the mode shape. The distribution of weld residual stress σ_x^r and σ_φ^r are shown in Fig. 5. Firstly, discuss the effect of weld residual stress amplitude on the mode shape. On the premise of maintaining the distribution (shown in Fig. 6), the amplitude is respectively equal to 100 % and 75 %. The typical profiles of several order mode shapes are shown in Fig. 9.



a) The first order mode shape



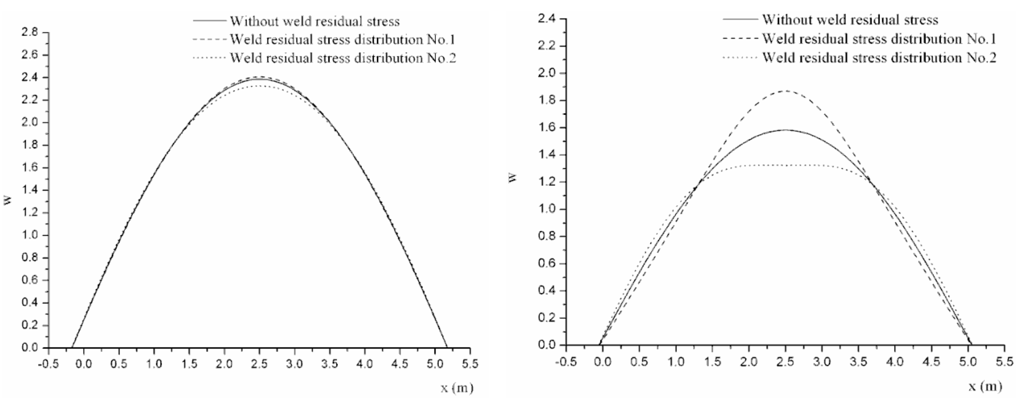
b) The third order mode shape

Fig. 9. Mode shapes of cylindrical shell with different weld residual stress amplitude at $\varphi = 0$

It can be found the mode shape of cylindrical shell without weld residual stress is single triangular function, and the mode shape with weld residual stress is the sum of several triangular functions. With the increase of amplitude, the variation of mode shape grows. So the amplitude mainly effects the variation range of mode shape.

Next the effect of weld residual stress distribution on the mode shape will be investigated. Two kinds of weld residual stress distribution which have the same amplitude are shown in Fig. 5 and Fig. 7. The typical profiles of several order mode shapes are shown in Fig. 10.

It can be found that different kind of weld residual stress distribution has different impact on the shape mode. Distribution No. 1 makes the middle of the mode shape sharper, whereas distribution No. 2 makes it smoother. So the distribution mainly has the influence on the variation trend of mode shape.



a) The first order mode shape b) The third order mode shape

Fig. 10. Mode shapes of cylindrical shell with different kind of weld residual stress distribution at $\varphi = 0$

5. Conclusions

The interaction between weld residual stress and displacements is investigated and derived here, and motion equations of cylindrical shell with weld residual stress are established based on Flügge theory. The analytical method is applied to calculate the vibrational mode. Weld residual stress can lead to the variation of free vibrational behavior. The amplitude has an important effect on the variation magnitude of natural frequency and mode shape, and the distribution does on the variation trend. So the effect of weld residual stress on vibrational characteristics of cylindrical shell can't be neglected.

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Appendix

$$\begin{aligned}\int_0^l &\sin(g\lambda x)\sin(\eta\lambda x)\cos(m\lambda x)dx\\ &=\frac{1}{2}\int_0^l\{\cos[(\eta-g)\lambda x]-\cos[(\eta+g)\lambda x]\}\cos(m\lambda x)dx\\ &=\frac{1}{2}\int_0^l\cos[(\eta-g)\lambda x]\cos(m\lambda x)\,dx-\frac{1}{2}\int_0^l\cos[(\eta+g)\lambda x]\cos(m\lambda x)dx,\end{aligned}\tag{A1}$$

$$\begin{aligned}\int_0^l &\cos(g\lambda x)\cos(\eta\lambda x)\cos(m\lambda x)dx\\ &=\frac{1}{2}\int_0^l\{\cos[(\eta+g)\lambda x]+\cos[(\eta-g)\lambda x]\}\cos(m\lambda x)dx\\ &=\frac{1}{2}\int_0^l\cos[(\eta+g)\lambda x]\cos(m\lambda x)\,dx+\frac{1}{2}\int_0^l\cos[(\eta-g)\lambda x]\cos(m\lambda x)dx,\end{aligned}\tag{A2}$$

$$\begin{aligned}\int_0^l &\cos(g\lambda x)\sin(\eta\lambda x)\sin(m\lambda x)dx\\ &=\int_0^l\frac{1}{2}\{\sin[(\eta+g)\lambda x]+\sin[(\eta-g)\lambda x]\}\sin(m\lambda x)dx\\ &=\int_0^l\frac{1}{2}\sin[(\eta+g)\lambda x]\sin(m\lambda x)\,dx+\int_0^l\frac{1}{2}\sin[(\eta-g)\lambda x]\sin(m\lambda x)dx,\end{aligned}\tag{A3}$$

$$\begin{aligned}\int_0^l &\sin(g\lambda x)\cos(\eta\lambda x)\sin(m\lambda x)dx\\ &=\frac{1}{2}\int_0^l\{\sin[(\eta+g)\lambda x]-\sin[(\eta-g)\lambda x]\}\sin(m\lambda x)dx\\ &=\frac{1}{2}\int_0^l\sin[(\eta+g)\lambda x]\sin(m\lambda x)\,dx-\frac{1}{2}\int_0^l\sin[(\eta-g)\lambda x]\sin(m\lambda x)dx,\end{aligned}\tag{A4}$$

$$\begin{aligned}\int_0^{2\pi}&\cos(j\varphi)\cos(\varsigma\varphi)\cos(\eta\varphi)d\varphi=\frac{1}{2}\int_0^{2\pi}\{\cos[(\varsigma+j)\varphi]+\cos[(\varsigma-j)\varphi]\}\cos(n\varphi)d\varphi\\ &=\frac{1}{2}\int_0^{2\pi}\cos[(\varsigma+j)\varphi]\cos(n\varphi)\,d\varphi+\frac{1}{2}\int_0^{2\pi}\cos[(\varsigma-j)\varphi]\cos(n\varphi)d\varphi,\end{aligned}\tag{A5}$$

$$\begin{aligned}\int_0^{2\pi}&\sin(j\varphi)\cos(\varsigma\varphi)\sin(n\varphi)d\varphi=\frac{1}{2}\int_0^{2\pi}\{\sin[(\varsigma+j)\varphi]-\sin[(\varsigma-j)\varphi]\}\sin(n\varphi)d\varphi\\ &=\frac{1}{2}\int_0^{2\pi}\sin[(\varsigma+j)\varphi]\sin(n\varphi)\,d\varphi-\frac{1}{2}\int_0^{2\pi}\sin[(\varsigma-j)\varphi]\sin(n\varphi)d\varphi,\end{aligned}\tag{A6}$$

$$\begin{aligned}\int_0^{2\pi}&\sin(j\varphi)\cos(\varsigma\varphi)\sin(n\varphi)d\varphi=\frac{1}{2}\int_0^{2\pi}\{\sin[(\varsigma+j)\varphi]-\sin[(\varsigma-j)\varphi]\}\sin(n\varphi)d\varphi\\ &=\frac{1}{2}\int_0^{2\pi}\sin[(\varsigma+j)\varphi]\sin(n\varphi)\,d\varphi-\frac{1}{2}\int_0^{2\pi}\sin[(\varsigma-j)\varphi]\sin(n\varphi)d\varphi,\end{aligned}\tag{A7}$$

$$\begin{aligned}\int_0^{2\pi}\sin(j\varphi)\sin(\varsigma\varphi)\cos(n\varphi)d\varphi &= \frac{1}{2}\int_0^{2\pi}\{\cos[(\varsigma+j)\varphi]-\cos[(\varsigma-j)\varphi]\}\cos(n\varphi)d\varphi \\ &= \frac{1}{2}\int_0^{2\pi}\cos[(\varsigma+j)\varphi]\cos(n\varphi)\,d\varphi - \frac{1}{2}\int_0^{2\pi}\cos[(\varsigma-j)\varphi]\cos(n\varphi)d\varphi,\end{aligned} \tag{A8}$$

$$\begin{aligned}\int_0^{2\pi}\cos(j\varphi)\sin(\varsigma\varphi)\sin(n\varphi)d\varphi &= \int_0^{2\pi}\frac{1}{2}\{\sin[(\varsigma+j)\varphi]+\sin[(\varsigma-j)\varphi]\}\sin(n\varphi)d\varphi \\ &= \frac{1}{2}\int_0^{2\pi}\sin[(\varsigma+j)\varphi]\sin(n\varphi)\,d\varphi + \frac{1}{2}\int_0^{2\pi}\sin[(\varsigma-j)\varphi]\sin(n\varphi)d\varphi,\end{aligned} \tag{A9}$$

$$\frac{1}{2}\int_0^{l}\sin[(\eta-g)\lambda x]\sin(m\lambda x)\,dx = \begin{cases} l/4, & m+g=\eta, \\ 0, & m+g\neq\eta, \end{cases} \tag{A10}$$

$$\frac{1}{2}\int_0^{l}\sin[(\eta+g)\lambda x]\sin(m\lambda x)\,dx = \begin{cases} l/4, & m-g=\eta, \\ -l/4, & m-g=-\eta, \\ 0, & |m-g|\neq\eta, \end{cases} \tag{A11}$$

$$\frac{1}{2}\int_0^{l}\cos[(\eta-g)\lambda x]\cos(m\lambda x)\,dx = \begin{cases} l/4, & m+g=\eta, \\ 0, & m+g\neq\eta, \end{cases} \tag{A12}$$

$$\frac{1}{2}\int_0^{l}\cos[(\eta+g)\lambda x]\cos(m\lambda x)\,dx = \begin{cases} l/4, & |m-g|=\eta, \\ 0, & |m-g|\neq\eta, \end{cases} \tag{A13}$$

$$\int_0^{2\pi}\sin[(\varsigma-j)\varphi]\sin(n\varphi)\,d\varphi = \begin{cases} \pi, & n+j=\varsigma, \\ 0, & n+j\neq\varsigma, \end{cases} \tag{A14}$$

$$\int_0^{2\pi}\sin[(\varsigma+j)\varphi]\sin(n\varphi)\,d\varphi = \begin{cases} \pi, & n-j=\varsigma, \\ -\pi, & n-j=-\varsigma, \\ 0, & |n-j|\neq\varsigma, \end{cases} \tag{A15}$$

$$\int_0^{2\pi}\cos[(\varsigma+j)\varphi]\cos(n\varphi)\,d\varphi = \begin{cases} \pi, & |n-j|=\varsigma, \\ 0, & |n-j|\neq\varsigma, \end{cases} \tag{A16}$$

$$\int_0^{2\pi}\cos[(\varsigma-j)\varphi]\cos(n\varphi)\,d\varphi = \begin{cases} \pi, & n+j=\varsigma, \\ 0, & n+j\neq\varsigma, \end{cases} \tag{A17}$$

$$\mathbf{\Lambda} = \begin{bmatrix} \mathbf{A}_1 & & & & \\ & \ddots & & & \\ & & \mathbf{A}_{(m-1)\times N+n} & & \\ & & & \ddots & \\ & & & & \mathbf{A}_{M\times N} \end{bmatrix}, \tag{A18}$$

where:

$$\mathbf{A}_m^n = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$a_{11} = -\frac{12\beta^2}{h^2} - \frac{6(1-\mu)n^2}{a^2h^2} - \frac{1-\mu}{2a^4}n^2 + \frac{\rho h\omega^2}{D}, \quad a_{12} = \frac{6(1+\mu)}{ah^2}n\beta,$$

$$a_{13} = \frac{12\mu\beta}{ah^2} + \frac{\beta^3}{a} - \frac{1-\mu}{2a^3}\beta n^2, \quad a_{21} = a_{12},$$

$$a_{22} = -\frac{12n^2}{a^2h^2} - \frac{6(1-\mu)}{h^2}\beta^2 - \frac{3(1-\mu)}{2a^2}\beta^2 + \frac{\rho h\omega^2}{D},$$

$$a_{23} = -\frac{12n}{a^2h^2} - \frac{3-\mu}{2a^2}\beta^2 n, \quad a_{31} = a_{13}, \quad a_{32} = a_{23},$$

$$a_{33} = -\frac{12}{a^2h^2} - \left(\beta^2 + \frac{n^2}{a^2}\right)^2 - \frac{1}{a^4} + 2\frac{n^2}{a^4} + \frac{\rho h\omega^2}{D}, \quad \beta = m\lambda.$$

$$\mathbf{R}_g=\begin{bmatrix}\mathbf{R}_{11}^g & \cdots & \mathbf{R}_{1q}^g \\ \vdots & & \vdots \\ \mathbf{R}_{p1}^g & \cdots & \mathbf{R}_{pq}^g\end{bmatrix}, \tag{A19}$$

where $\mathbf{R}_{pq}^g=\frac{h}{2D}\begin{bmatrix}R_{11}^{pq} & 0 & 0 \\ 0 & R_{22}^{pq} & R_{23}^{pq} \\ 0 & R_{32}^{pq} & R_{33}^{pq}\end{bmatrix}$, nonzero elements are:

When $p=m$, $q=m-g$, and $m-g>0$:

$$R_{11}^{pq}=-(m-g)m\lambda^2\sigma_g^{rx}-\frac{n^2}{a^2}\sigma_g^{r\varphi},\ R_{22}^{pq}=R_{33}^{pq}=-R_{11}^{pq},\ R_{23}^{pq}=\frac{n}{a^2}\sigma_g^{r\varphi},\ R_{32}^{pq}=R_{23}^{pq}.$$

When $p=m$, $q=|m-g|$, and $m-g<0$:

$$R_{11}^{pq}=-(m-g)m\lambda^2\sigma_g^{rx}-\frac{n^2}{a^2}\sigma_g^{r\varphi},\ R_{22}^{pq}=R_{33}^{pq}=R_{11}^{pq},\quad R_{23}^{pq}=-\frac{n}{a^2}\sigma_g^{r\varphi},\quad R_{32}^{pq}=\frac{n}{a^2}\sigma_g^{r\varphi}.$$

When $p=m$, $q=m+g$:

$$R_{11}^{pq}=-(m+g)m\lambda^2\sigma_g^{rx}-\frac{n^2}{a^2}\sigma_g^{r\varphi},\quad R_{22}^{pq}=R_{33}^{pq}=R_{11}^{pq},\quad R_{23}^{pq}=-\frac{n}{a^2}\sigma_g^{r\varphi},\quad R_{32}^{pq}=R_{23}^{pq}.$$

$$\mathbf{R}_{gj}=\begin{bmatrix}\mathbf{R}_{11}^{gj} & \cdots & \mathbf{R}_{1q}^{gj} \\ \vdots & & \vdots \\ \mathbf{R}_{p1}^{gj} & \cdots & \mathbf{R}_{pq}^{gj}\end{bmatrix}, \tag{A20}$$

where $\mathbf{R}_{pq}^{gj}=\frac{h}{4D}\begin{bmatrix}R_{11}^{pq} & 0 & 0 \\ 0 & R_{22}^{pq} & R_{23}^{pq} \\ 0 & R_{32}^{pq} & R_{33}^{pq}\end{bmatrix}$, nonzero elements are:

When $p=(m-1)\times N+n$, $q=(m-g)\times N+(n-j+1)$, $m-g>0$ and $n-j>0$:

$$R_{11}^{pq}=-(m-g)m\lambda^2\sigma_{gj}^{rx}-\frac{(n-j)n}{a^2}\sigma_{gj}^{r\varphi},\ R_{22}^{pq}=R_{11}^{pq},\quad R_{23}^{pq}=-\frac{n}{a^2}\sigma_{gj}^{r\varphi},$$
$$R_{32}^{pq}=R_{23}^{pq},\ R_{33}^{pq}=R_{11}^{pq}.$$

When $p=(m-1)\times N+n$, $q=(m-g)\times N+|n-j|$, $m-g>0$ and $n-j<0$:

$$R_{11}^{pq}=-(m-g)m\lambda^2\sigma_{gj}^{rx}-\frac{(n-j)n}{a^2}\sigma_{gj}^{r\varphi},\ R_{22}^{pq}=-R_{11}^{pq},\ R_{23}^{pq}=\frac{n}{a^2}\sigma_{gj}^{r\varphi},$$
$$R_{32}^{pq}=-R_{23}^{pq},\ R_{33}^{pq}=R_{11}^{pq}.$$

When $p=(m-1)\times N+n$, $q=|m-g|\times N+(n-j+1)$, $m-g<0$ and $n-j>0$:

$$R_{11}^{pq}=-(m-g)m\lambda^2\sigma_{gj}^{rx}-\frac{(n-j)n}{a^2}\sigma_{gj}^{r\varphi},\ R_{22}^{pq}=-R_{11}^{pq},\ R_{23}^{pq}=\frac{n}{a^2}\sigma_{gj}^{r\varphi},$$
$$R_{32}^{pq}=R_{23}^{pq},\ R_{33}^{pq}=-R_{11}^{pq}.$$

When $p=(m-1)\times N+n$, $q=|m-g|\times N+(|n-j|+1)$, $m-g<0$ and $n-j<0$:

$$R_{11}^{pq}=-(m-g)m\lambda^2\sigma_{gj}^{rx}-\frac{(n-j)n}{a^2}\sigma_{gj}^{r\varphi},\ \ R_{22}^{pq}=R_{11}^{pq},\ \ R_{23}^{pq}=-\frac{n}{a^2}\sigma_{gj}^{r\varphi},$$
$$R_{32}^{pq}=-R_{23}^{pq},\ \ R_{33}^{pq}=-R_{11}^{pq}.$$

When $p=(m-1)\times N+n, q=(m+g)\times N+n-j, n-j>0$:

$$R_{11}^{pq}=-(m+g)m\lambda^2\sigma_{gj}^{rx}-\frac{(n-j)n}{a^2}\sigma_{gj}^{r\varphi},\ \ R_{22}^{pq}=R_{11}^{pq},\ \ R_{23}^{pq}=-\frac{n}{a^2}\sigma_{gj}^{r\varphi},$$
$$R_{32}^{pq}=R_{23}^{pq},\ \ R_{33}^{pq}=R_{11}^{pq}.$$

When $p=(m-1)\times N+n, q=m+g\times N+|n-j|, n-j<0$:

$$R_{11}^{pq}=-(m+g)m\lambda^2\sigma_{gj}^{rx}-\frac{(n-j)n}{a^2}\sigma_{gj}^{r\varphi},\ \ R_{22}^{pq}=-R_{11}^{pq},\ \ R_{23}^{pq}=\frac{n}{a^2}\sigma_{gj}^{r\varphi},$$
$$R_{32}^{pq}=-R_{23}^{pq},\ \ R_{33}^{pq}=R_{11}^{pq}.$$

When $p=(m-1)\times N+n, q=(m-g)\times N+n+j, m-g>0$:

$$R_{11}^{pq}=-(m-g)m\lambda^2\sigma_{gj}^{rx}-\frac{(n+j)n}{a^2}\sigma_{gj}^{r\varphi},\ \ R_{22}^{pq}=R_{11}^{pq},\ \ R_{23}^{pq}=-\frac{n}{a^2}\sigma_{gj}^{r\varphi},$$
$$R_{33}^{pq}=R_{11}^{pq},\ \ R_{32}^{pq}=R_{23}^{pq}.$$

When $p=(m-1)\times N+n, q=|m-g|\times N+n+j, m-g<0$:

$$R_{11}^{pq}=-(m-g)m\lambda^2\sigma_{gj}^{rx}-\frac{(n+j)n}{a^2}\sigma_{gj}^{r\varphi},\ \ R_{22}^{pq}=-R_{11}^{pq},\ \ R_{23}^{pq}=\frac{n}{a^2}\sigma_{gj}^{r\varphi},$$
$$R_{32}^{pq}=R_{23}^{pq},\ \ R_{33}^{pq}=-R_{11}^{pq}.$$

When $p=(m-1)\times N+n, q=(m+g)\times N+n+j$:

$$R_{11}^{pq}=-(m+g)m\lambda^2\sigma_{gj}^{rx}-\frac{(n+j)n}{a^2}\sigma_{gj}^{r\varphi},\ \ R_{22}^{pq}=R_{11}^{pq},\ \ R_{23}^{pq}=-\frac{n}{a^2}\sigma_{gj}^{r\varphi},$$
$$R_{32}^{pq}=R_{23}^{pq},\ \ R_{33}^{pq}=R_{11}^{pq}.$$

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